RICCI FLOW DEFORMATION OF COSMOLOGICAL INITIAL DATA SETS

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Ricci flow deformation of cosmological initial data sets in general relativity is a technique for generating families of initial data sets which potentially would allow to interpolate between distinct spacetimes. This idea has been around since the appearance of the Ricci flow on the scene, but it has been difficult to turn it into a sound mathematical procedure. In this expository talk we illustrate, how Perelman's recent results in Ricci flow theory can considerably improve on such a situation. From a physical point of view this analysis can be related to the issue of finding a constant–curvature template spacetime for the inhomogeneous Universe, relevant to the interpretation of observational data and, hence, bears relevance to the dark energy and dark matter debates. These techniques provide control on curvature fluctuations (intrinsic backreaction terms) in their relation to the averaged matter distribution.

Keywords: Ricci flow, Relativistic cosmology, initial–value problem in GR.

1. INTRODUCTION

The Ricci flow has been introduced by R. Hamilton[19] with the goal of providing an analytic approach to Thurston's geometrization conjecture for three-manifolds[30,31]. Inspired by the theory of harmonic maps, he considered the geometric evolution equation obtained when one evolves a Riemannian metric g_{ab}, on a three-manifold Σ, in the direction of its Ricci tensor[20]

$\mathcal{R}_{ab}$, *i.e.*

$$\begin{cases} \frac{\partial}{\partial \beta} g_{ab}(\beta) = -2\, \mathcal{R}_{ab}(\beta)\ , \\ \\ g_{ab}(\beta = 0) = g_{ab}\ \ ,\ \ 0 \leq \beta < T_0\ . \end{cases} \tag{1}$$

In recent years, this geometric flow has gained extreme popularity thanks to the revolutionary breakthroughs of G. Perelman[25–27] , who, taking the whole subject by storm, has brought to completion Hamilton's approach to Thurston's conjecture. The prominent themes recurring in Hamilton's and Perelman's works converge to a proof that the Ricci flow, coupled to topological surgery, provides a natural technique for factorizing and uniformizing a three-dimensional Riemannian manifold (Σ, g) into locally homogeneous geometries. This is a result of vast potential use also in theoretical physics, where the Ricci flow often appears in disguise as a natural real-space renormalization group flow. Non-linear σ-model theory, describing quantum strings propagating in a background spacetime, affords the standard case study in such a setting[1,2,8,16,23,24] .

Another paradigmatical, perhaps even more direct, application occurs in relativistic cosmology[9,10] , (for a series of recent results see also[6,7] and the references cited therein). This will be related to the main topic of this talk, and to motivate our interest in it, let us recall that homogeneous and isotropic solutions of Einstein's laws of gravity (the Friedman–Lemaitre–Robertson–Walker (FLRW) spacetimes) do not account for inhomogeneities in the Universe. The question whether they do *on average* is an issue[14] that is the subject of considerable debate especially in the recent literature (see[21,28] and follow–up references; comprehensive lists may be found in[15,29] and[5]).

In any case, a member of the family of FLRW cosmologies (the so–called concordance model that is characterized by a dominating cosmological constant in a spatially flat universe model) provides a successful *fitting model* to a large number of observational data, and the generally held view is that the spatial sections of FLRW spacetimes indeed describe the *physical Universe* on a sufficiently large averaging scale. This raises an interesting problem in mathematical cosmology: devise a way to explicitly construct a constant–curvature metric out of a *scale–dependent* inhomogeneous distribution of matter and spatial curvature. It is in such a framework that one makes the basic observation that the Ricci flow (1) and its linearization, provide a natural technique[6,10] for deforming, and under suitable conditions smoothing, the geometrical part of scale–dependent cosmological initial data sets. Moreover, by taking advantage of some elementary aspects of Perelman's

results, this technique also provides a natural and unique way for deforming, along the Ricci flow, the matter distribution. The expectation is that in this way we can define a deformation of cosmological initial data sets into a one-parameter family of initial data whose time evolution, along the evolutive part of Einstein's equations, describe the Ricci flow deformation of a cosmological spacetime.

2. The kinematical set–up: Initial data set for cosmological spacetimes

To set notation, we emphasize that throughout the paper we shall consider a smooth three-dimensional manifold Σ, which we assume to be closed and without boundary. We let $C^\infty(\Sigma,\mathbb{R})$ and $C^\infty(\Sigma,\otimes^p T^*\Sigma\otimes^q T\Sigma)$ be the space of smooth functions and of smooth (p,q)–tensor fields on Σ, respectively.

We shall denote by $\mathcal{D}iff(\Sigma)$ the group of smooth diffeomorphisms of Σ, and by $\mathcal{R}iem(\Sigma)$ the space of all smooth Riemannian metrics over Σ. The tangent space , $\mathcal{T}_{(\Sigma,g)}\mathcal{R}iem(\Sigma)$, to $\mathcal{R}iem(\Sigma)$ at (Σ,g) can be naturally identified with the space of symmetric bilinear forms $C^\infty(\Sigma,\otimes^2 T^*\Sigma)$ over Σ. The hypothesis of smoothness has been made for simplicity. Results similar to those described below, can be obtained for initial data sets with finite Holder or Sobolev differentiability. In such a framework, let us recall that a collection of fields $g\in\mathcal{R}iem(\Sigma)$, $K\in\mathcal{T}_{(\Sigma,g)}\mathcal{R}iem(\Sigma)$, $\varrho\in C^\infty(\Sigma,\mathbb{R}^+)$, $\vec{J}\in C^\infty(\Sigma,T\Sigma)$, defined over the three-manifold Σ, characterizes a set $(\ \Sigma\ ,\ g_{ab}\ ,\ K_{ab}\ ,\ \varrho\ ,\ J_a\ )$, of physical cosmological initial data for Einstein equations if and only if the *matter fields* $(\varrho,\vec{J})$ verify the weak energy condition $\varrho\geq 0$, the dominant energy condition $\varrho^2\geq g_{ab}J^aJ^b$, and their coupling with the geometric fields (g,K) is such as to satisfy the Hamiltonian and divergence constraints[a]:

$$\mathcal{R}+k^2-K^a{}_bK^b{}_a=16\pi G\varrho+2\Lambda\ , \tag{2}$$

$$\nabla_bK^b{}_a-\nabla_ak=8\pi GJ_a\ . \tag{3}$$

Here Λ is the cosmological constant, $k:=g^{ab}K_{ab}$, and $\mathcal{R}$ is the scalar curvature of the Riemannian metric g_{ab}. If such a set of admissible data is propagated according to the evolutive part of Einstein's equations, then the symmetric tensor field K_{ab} can be interpreted as the extrinsic curvature and k as the mean curvature of the embedding $i_t:\Sigma\to M^{(4)}$ of

[a]Latin indices run through $1,2,3$; we adopt the summation convention. The nabla operator denotes covariant derivative with respect to the 3–metric. The units are such that $c=1$.

(Σ, g_{ab}) in the spacetime $(M^{(4)} \simeq \Sigma \times \mathbb{R}, g^{(4)})$ resulting from the evolution of $(\Sigma, g_{ab}, K_{ab}, \varrho, J_a)$, whereas ϱ and J_a are, respectively, identified with the mass density and the momentum density of the material self–gravitating sources on (Σ, g_{ab}).

3. The Heuristics of averaging: Deformation of cosmological initial data sets

The averaging procedure described in[6,9,10] is based on a smooth deformation of the physical initial data $(\Sigma, g_{ab}, K_{ab}, \varrho, J_a)$ into a one-parameter family of initial data sets

$$\beta \longmapsto (\Sigma \ , \ g_{ab}(\beta) \ , \ K_{ab}(\beta) \ , \ \varrho(\beta) \ , \ J_a(\beta)), \tag{4}$$

with $0 \leq \beta \leq \infty$ being a parameter characterizing the averaging scale. The general idea is to construct the flow (4) in such a way as to represent, as β increases, a scale-dependent averaging of $(\Sigma, g_{ab}, K_{ab}, \varrho, J_a)$, and – under suitable hypotheses – reducing it to a constant-curvature initial data set

$$(\Sigma \ , \ \overline{g}_{ab} \ , \ \overline{K}_{ab} = \frac{1}{3}\overline{g}_{ab}\overline{k} \ , \ \overline{\varrho} \ , \ \overline{J}_a = 0), \tag{5}$$

where $\overline{g}_{ab}$ is a constant curvature metric on Σ, $\overline{k}$ is the (spatially constant) trace of the extrinsic curvature (related to the Hubble parameter), and $\overline{\varrho}$ is the averaged matter density. Under the heading of such a general strategy it is easy to figure out the reasons for an important role played by the Ricci flow and its linearization. As we shall recall shortly, they are natural geometrical flows always defining a non-trivial deformation of the metric g_{ab} and of the extrinsic curvature K_{ab}. Moreover, when global, they posses remarkable smoothing properties. For instance, if, for $\beta = 0$, the scalar curvature $\mathcal{R}$ of (Σ, g_{ab}) is > 0, and if there exist positive constants α_1, α_2, α_3, not depending on β, such that $\mathcal{R}_{ab}(\beta) - \alpha_1 g_{ab}(\beta)\mathcal{R}(\beta) \geq 0$, and $\widehat{\mathcal{R}}_{ab}(\beta)\widehat{\mathcal{R}}^{ab}(\beta) \leq \alpha_2 \mathcal{R}^{1-\alpha_3}(\beta)$, where $\widehat{\mathcal{R}}_{ab}(\beta) \doteq \mathcal{R}_{ab}(\beta) - \frac{1}{3}g_{ab}(\beta)\mathcal{R}(\beta)$ is the trace–free part of the Ricci tensor, then[19] the solutions $(g_{ab}(\beta), K_{ab}(\beta))$ of the (volume–normalized) Ricci flow and its linearization exist for all $\beta > 0$, and the pair $(g_{ab}(\beta), K_{ab}(\beta))$, uniformly converges, when $\beta \rightarrow \infty$, to $(\overline{g}_{ab}, \mathcal{L}_{\vec{v}}\overline{g}_{ab})$ where $\overline{g}_{ab}$ is a metric with constant positive sectional curvature, and $\vec{v}$ is some vector field on Σ, possibly depending on β. The flow $\beta \mapsto (\overline{g}_{ab}, \mathcal{L}_{\vec{v}}\overline{g}_{ab})$, describing a motion by diffeomorphisms over a constant curvature manifold, can be thought of as representing the smoothing of the geometrical part of an initial data set $(\Sigma, g_{ab}, K_{ab}, \varrho, J_a)$.

It is useful to keep in mind what we can expect and what we cannot expect out of such a Ricci–flow deformation of cosmological initial data set. Let us start by remarking that the family of data (4) will correspond to the initial data for physical spacetimes iff the constraints (2) and (3) hold throughout the β–dependent deformation. This is a very strong requirement and, if we have a technically consistent way $\beta \mapsto (\varrho(\beta), g_{ab}(\beta), K_{ab}(\beta))$ of deforming the matter distribution and the geometrical data, then the most natural way of implementing the constraints is to use them to define scale–dependent backreaction fields $\beta \mapsto \phi(\beta)$, $\beta \mapsto \psi_a(\beta)$ describing the non–linear interaction between matter averaging and geometrical averaging, *i.e.*,

$$\phi(\beta) \doteq \varrho(\beta) - (16\pi G)^{-1}\left[\mathcal{R}(\beta) + k^2(\beta) - K^a{}_b(\beta) K^b{}_a(\beta) - 2\Lambda\right]\ , \quad (6)$$

$$\psi_a(\beta) \doteq J_a(\beta) - (8\pi G)^{-1}\left[\nabla_b K^b{}_a(\beta) - \nabla_a k(\beta)\right]\ . \quad (7)$$

To illustrate how this strategy works, let us concentrate, in this talk, on the characterization of the scalar field $\beta \mapsto \phi(\beta)$, providing the backreaction between matter and geometrical averaging. The covector field $\beta \mapsto \psi_a(\beta)$ can, in principle, be controlled by the action of a β–dependent diffeomorphism. However, its analysis requires a subtle interplay with the kinematics of spacetime foliations[4] , (*i.e.*, how we deal with the lapse function and with the shift vector field in the framework of Perelman's approach), and will be discussed elsewhere, (for a pre–Perelman approach to this issue see[6,10]) .

Let us start by observing that the matter averaging flow $\beta \mapsto \varrho(\beta)$ must comply with the preservation of the physical matter content

$$\int_\Sigma \varrho(\beta)\, d\mu_{g(\beta)} = \int_\Sigma \varrho(\beta = 0)\, d\mu_{g(\beta=0)} \doteq M\ ,\ \ \forall\beta\ , \quad (8)$$

and must be explicitly coupled to the scale of geometrical averaging. In other words, if, for some fixed $\beta > 0$, we consider that part of the matter distribution $\varrho(\beta)$ which is localized in a given region $B(x, \tau(\beta)) \subset \Sigma_\beta$ of size $\tau(\beta)$, then we should be able to tell from which localized distribution $(\varrho_m, B(x, \tau))$, at $\beta = 0$, the selected matter content $(\varrho(\beta), B(x, \tau(\beta)))$ has evolved. A natural answer to these requirements is provided by Perelman's backward localization[25] of probability measures on Ricci evolving manifolds. The idea is to probe the Ricci flow with a probability measure whose dynamics can localize the regions of the manifold (Σ, g) of geometric interest. This is achieved by considering, along the solution $g_{ab}(\beta)$ of (1), a β–dependent mapping $\beta \longmapsto f(\beta,\ ) \in C^\infty(\Sigma_\beta, \mathbb{R})$, in terms of which one constructs on Σ_β the measure $d\varpi(\beta) \doteq (4\pi\tau(\beta))^{-\frac{3}{2}}\, e^{-f} d\mu_{g(\beta)}$,

where $\beta \longmapsto \tau(\beta) \in \mathbb{R}^+$ is a scale parameter chosen in such a way as to normalize $d\varpi(\beta)$ according to the so–called *Perelman's coupling*: $\int_{\Sigma_\beta} d\varpi(\beta) = (4\pi\tau(\beta))^{-\frac{3}{2}} \int_{\Sigma_\beta} e^{-f} d\mu_{g(\beta)} = 1$. It is easily verified that this is preserved in form along the Ricci flow (1), if the mapping f and the scale parameter $\tau(\beta)$ are evolved backward in time $\beta \in (\beta^*, 0)$ according to the coupled flows defined by

$$\begin{cases} \frac{\partial}{\partial\beta} f = -\Delta_{g(\beta)} f - R(\beta) + \frac{3}{2}\tau(\beta)^{-1},\ f(\beta^*) = f_0 \\ \\ \frac{d}{d\beta}\,\tau(\beta) = -1,\ \tau(\beta^*) = \tau_0, \end{cases} \tag{9}$$

where $\Delta_{g(\beta)}$ is the Laplacian with respect to the metric $g_{ab}(\beta)$, and f_0, τ_0 are given (final) data. In this connection, note that the equation for f is a backward heat equation, and as such the forward evolution $f(\beta = 0) \rightarrow f$ is an ill-posed problem. A direct way for circumventing such a difficulty is to interpret (9) according to the following two-steps prescription: (i) Evolve the metric $\beta \longmapsto (\Sigma, g_{ab}(\beta))$, say up to some β^*, according to the Ricci flow, (if the flow is global we may let $\beta^* \rightarrow \infty$); (ii) On the Ricci evolved Riemannian manifold $(\Sigma, \overline{g}_{ab}(\beta^*))$ so obtained, select a function $f(\beta^*)$ and the corresponding scale parameter $\tau(\beta^*)$, and evolve them, backward in β, according to (9).

4. Ricci–flow deformation of the initial data $(\Sigma, g_{ab}, K_{ab}, \varrho)$

With these preliminary remarks along the way, let us characterize the various steps involved in constructing the flow (4). (For ease of exposition, we refer to the standard unnormalized flow; volume normalization can be enforced by a reparametrization of the deformation parameter).

Definition 4.1. (*Geometrical Data Deformation*) Given an initial data set $(\Sigma, g_{ab}, K_{ab}, \varrho, J_a)$ for a cosmological spacetime $(M^{(4)} \simeq \Sigma \times \mathbb{R}, g^{(4)})$, the Ricci–flow deformation of its geometrical part (Σ, g_{ab}, K_{ab}) is defined by the flow $\beta \mapsto (g_{ab}(\beta), K_{hj}(\beta))$, $0 \leq \beta < \beta^*$ provided by the (weakly) parabolic initial value problem (the Ricci flow, proper)

$$\begin{cases} \frac{\partial}{\partial\beta} g_{ab}(\beta) = -2\mathcal{R}_{ab}(\beta), \\ \\ g_{ab}(\beta = 0) = g_{ab}\ \ ,\ \ 0 \leq \beta <\ \beta^* \\ \\ \lim_{\beta \nearrow \beta^*} \left[\sup_{x\in\Sigma} |Rm(x,\beta)|\right] < \infty\ , \end{cases} \tag{10}$$

and by its linearization, which, by suitably fixing the action of the diffeomorphism group $\mathcal{D}iff(\Sigma)$, takes the form of the parabolic initial value problem[13]

$$\begin{cases} \frac{\partial}{\partial \beta} K_{ab}(\beta) = \Delta_L \, K_{ab}(\beta) \ , \\ \\ K_{ab}(\beta = 0) = \ K_{ab} \ \ , \quad 0 \leq \beta < \beta^* \ , \end{cases} \tag{11}$$

where Δ_L denotes the Lichnerowicz–DeRham Laplacian $\Delta_L K_{ab} \doteq \nabla^i \nabla_i K_{ab} - R_{as} K^s_b - R_{bs} K^s_a + 2 R_{asbt} K^{st}$ acting on symmetric bilinear forms[22] .

Definition 4.2. (*Localization of the Deformed Data*) The geometrical deformation $\beta \mapsto (g_{ab}(\beta), K_{hj}(\beta))$ is controlled by the backward localizing flow $\eta \mapsto (E(x, y; \eta), E^{ab}_{i'k'}(x, y; \eta))$, $\eta \doteq \beta^* - \beta$, defined by the backward heat kernels for the conjugate[11,12] parabolic initial value problem associated with the $g(\eta)$–dependent Laplace-Beltrami and Lichnerowicz operators $\eta \mapsto (\Delta, \, \Delta_L)$. The operator Δ_L, when acting on (bi)scalars, reduces to Δ. Thus, we can characterize both these kernels in a compact form as solutions of

$$\begin{cases} \frac{\partial}{\partial \eta} \begin{pmatrix} E^{ab}_{i'k'}(x, y; \eta) \\ E(x, y; \eta) \end{pmatrix} = \Delta^{(x)}_L \begin{pmatrix} E^{ab}_{i'k'}(x, y; \eta) \\ E(x, y; \eta) \end{pmatrix} - \mathcal{R} \begin{pmatrix} E^{ab}_{i'k'}(x, y; \eta) \\ E(x, y; \eta) \end{pmatrix} , \\ \\ \lim_{\, \eta \searrow 0^+} \begin{pmatrix} E^{ab}_{i'k'}(x, y; \eta) \\ E(x, y; \eta) \end{pmatrix} = \begin{pmatrix} \delta^{ab}_{i'k'}(x, y; \eta) \\ \delta(x, y; \eta) \end{pmatrix} , \end{cases} \tag{12}$$

where $(y, x; \eta) \in (\Sigma \times \Sigma \backslash Diag(\Sigma \times \Sigma)) \times [0, \beta^*]$, $\Delta^{(x)}_L$ denotes the Lichnerowicz–DeRham Laplacian with respect to the variable x, the heat kernels $E^{ab}_{i'k'}(y, x; \eta)$ and $E(x, y; \eta)$ are smooth sections of $(\otimes^2 T\Sigma) \boxtimes (\otimes^2 T^*\Sigma)$ and $\Sigma \boxtimes \Sigma$, respectively, and finally, $\delta^{k_1 \ldots k_p}_{i'_1 \ldots i'_p}(y, x; \eta)$ is the Dirac p–tensorial measure, $(p = 0, 1, \ldots)$ on $(\Sigma, g(\eta))$. The Dirac initial condition is understood in the distributional sense, *i.e.*, $\int_{\Sigma_\eta} K^{ab}_{i'k'}(y, x; \eta) \; w^{i'k'}(y) \; d\mu^{(y)}_{g(\eta)} \to w^{ab}(x) \quad as \quad \eta \searrow 0^+$, for any smooth symmetric bilinear form with compact support $w^{i'k'} \in C^\infty_0(\Sigma, \otimes^2 T\Sigma)$, and where the limit is meant in the uniform norm on $C^\infty_0(\Sigma, \otimes^2 T\Sigma)$.

Note that heat kernels for generalized Laplacians, such as Δ_L, (smoothly) depending on a one–parameter family of metrics $\varepsilon \mapsto g_{ab}(\varepsilon)$, $\varepsilon \geq 0$, are briefly dealt with in[3] . The delicate setting where the parameter dependence is, as in our case, identified with the parabolic time driving the diffusion of the kernel, is discussed in[13,18] , (see Appendix A, §7 for a characterization

of the parametrix of the heat kernel in such a case), and in[17]. Strictly speaking, in all these works, the analysis is confined to the scalar Laplacian, possibly with a potential term, but the theory readily extends to generalized Laplacians, always under the assumption that the metric $g_{ab}(\beta)$ is smooth as $\nearrow \beta^*$. Finally, the kernels for Δ and Δ_L can both be normalized, along the Ricci flow, over the round (collapsing) 3–sphere $(\mathbb{S}^3, g_{can}(\beta))$.

Let us now consider the matter content localized by taking the $d\varpi(\eta)$–expectation of $\varrho(\eta)$,

$$\int_\Sigma \varrho(\eta)\, d\varpi(\eta) \doteq M(d\varpi(\eta)) \ . \tag{13}$$

According to (8), we require that such a local mass is preserved along the η–evolution of the measure $d\varpi(\eta)$, *i.e.*, $\frac{d}{d\beta}\int_\Sigma \varrho(\eta)\, d\varpi(\eta) = 0$. This request motivates the following

Definition 4.3. (*Deformation of Matter Data*) The given matter distribution $\varrho(\beta = 0)$ is deformed according to the heat–flow $\beta \mapsto \varrho(\beta)$ given by

$$\begin{cases} \frac{\partial}{\partial\beta}\varrho(\beta) = \Delta_{g(\beta)}\ \varrho(\beta),\ \ \beta \in [0, T_0) \\ \\ \varrho(\beta = 0) = \varrho. \end{cases} \tag{14}$$

We are now in the position to characterize the Ricci flow deformation of the cosmological data $(\Sigma, g_{ab}, K_{ab}, \varrho, J_a)$. According to the results described in[11] we have

Proposition 4.1. *Let $\beta \mapsto (g_{ab}(\beta), K_{ik}(\beta), \varrho(\beta))$ be the Ricci flow deformation, on $\Sigma_\eta \times [0, \beta^*]$, of the data $(\Sigma, g_{ab}, K_{ab}, \varrho)$ as defined above. Assume that the underlying Ricci flow $\beta \mapsto (\Sigma, g_{ab}(\beta))$ is of bounded geometry, and let $E(y, x; \eta)$ and $E^{ab}_{i'k'}(y, x; \eta)$ be the (backward) heat kernels Ricci-flow conjugated to the Laplace–Beltrami and to the Lichnerowicz–DeRham operator, respectively. Then, for all $0 \leq \eta \leq \beta^*$,*

$$g_{i'k'}\,(y, \eta = 0) = \int_\Sigma E^{ab}_{i'k'}(y, x; \eta)\ [g_{ab}(x, \eta) - 2\eta\ \mathcal{R}_{ab}(x, \eta)]\ d\mu_{g(x,\eta)} \ , \tag{15}$$

$$\begin{pmatrix} \mathcal{R}_{i'k'}(y, \eta = 0) \\ K_{i'k'}(y, \eta = 0) \end{pmatrix} = \int_\Sigma E^{ab}_{i'k'}(y, x; \eta) \begin{pmatrix} \mathcal{R}_{ab}(x, \eta) \\ K_{ab}(x, \eta) \end{pmatrix} d\mu_{g(x,\eta)} \ , \tag{16}$$

$$\textit{and} \qquad \varrho(y, \eta = 0) = \int_\Sigma E(y, x; \eta)\, \varrho(x, \eta)\, d\mu_{g(x,\eta)} \ . \tag{17}$$

Moreover, as $\eta \searrow 0^+$, we have the uniform asymptotic expansion

$$\begin{aligned}
&\begin{pmatrix} \mathcal{R}_{i'k'}(y,\eta=0) \\ K_{i'k'}(y,\eta=0) \end{pmatrix} = \\
&\frac{1}{(4\pi\,\eta)^{\frac{3}{2}}} \int_\Sigma \exp\left(-\frac{d_0^2(y,x)}{4\eta}\right) \tau_{i'k'}^{ab}(y,x;\eta) \begin{pmatrix} \mathcal{R}_{ab}(x,\eta) \\ K_{ab}(x,\eta) \end{pmatrix} d\mu_{g(x,\eta)} \\
&+\sum_{h=1}^{N} \frac{\eta^h}{(4\pi\,\eta)^{\frac{3}{2}}} \int_\Sigma \exp\left(-\frac{d_0^2(y,x)}{4\eta}\right) \Phi[h]_{i'k'}^{ab}(y,x;\eta) \begin{pmatrix} \mathcal{R}_{ab}(x,\eta) \\ K_{ab}(x,\eta) \end{pmatrix} d\mu_{g(x,\eta)} \\
&+O\left(\eta^{N-\frac{1}{2}}\right) ,
\end{aligned} \tag{18}$$

where $\tau_{i'k'}^{ab}(y,x;\eta) \in T\Sigma_\eta \boxtimes T^*\Sigma_\eta$ is the parallel transport operator associated with $(\Sigma, g(\eta))$, $d_0(y,x)$ is the distance function in $(\Sigma, g(\eta=0))$, and $\Phi[h]_{i'k'}^{ab}(y,x;\eta)$ are smooth sections $\in C^\infty(\Sigma\times\Sigma', \otimes^2 T\Sigma \boxtimes \otimes^2 T^*\Sigma)$, (depending on the geometry of $(\Sigma, g(\eta))$), characterizing the asymptotics of the heat kernel $E_{i'k'}^{ab}(y,x;\eta)$. With an obvious adaptation, such an asymptotic behavior also extends to $g_{i'k'}(y,\eta=0)$ and $\varrho(y,\eta=0)$.

With these results it is rather straightforward to provide useful characterizations of the (intrinsic) backreaction field $\phi(\beta)$. For illustrative purposes, let us assume that $\beta \mapsto K_{ab}(\beta) \equiv 0$ and $\Lambda \equiv 0$, then from (6) and the above proposition we get that

$$\begin{aligned}
\phi(y,\eta=0) = &\int_\Sigma E(y,x;\eta)\,\varrho(x,\eta)\,d\mu_{g(x,\eta)} \\
&-(16\pi\,G)^{-1}\left[\int_\Sigma g^{i'k'}(y,\eta=0)\,E_{i'k'}^{ab}(y,x;\eta)\,\mathcal{R}_{ab}(x,\eta)\,d\mu_{g(x,\eta)}\right] .
\end{aligned} \tag{19}$$

If we further assume that the Hamiltonian constraint holds at the *fixed* observative scale η and that $\mathcal{R}_{ab}(x,\eta) \approx [\frac{1}{3}\mathcal{R}(\eta)\,g_{ab}(x,\eta=0) + \delta\mathcal{R}_{ab}(x,\eta)]$, (*i.e.*, curvature fluctuates around the *constant curvature background* $\mathcal{R}_{l'm'}(y,\eta=0) = 2C\,g_{l'm'}(y,\eta=0)$), then

$$\phi(y,\eta=0) \approx (16\pi\,G)^{-1}\int_\Sigma g^{i'k'}(y,\eta=0)\,E_{i'k'}^{ab}(y,x;\eta)\,\delta\mathcal{R}_{ab}(x,\eta)\,d\mu_{g(x,\eta)} , \tag{20}$$

from which it follows that the intrinsic backreaction field $\phi(\beta)$ is generated by curvature fluctuations around the given background, as expected.

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